Manifolds

Eric Errthum & Luke Gutzwiller

Imagine you're at a picnic. You've brought along, for whatever reasons, a coffee cup, a doughnut, a pumpkin, and a Klein bottle someone tried (and failed) to fill with lemonade. Imagine, now, that an ant or two has stumbled across your pastoral feast, and starts to crawl here and there over your various items. Clearly, to our eyes, a coffee cup and a pumpkin are not terribly similar. What might they look like to the ant, though? The ant can't see the cup, the pumpkin, the doughnut, or the bottle all at once. It can only see tiny sections of them at a time, and, assuming our ant is colorblind, they all look more or less alike. The facts that two have holes in them and that one has only one side are invisible at this scale. The ant probably sees itself as hiking along a more or less flat and featureless plain. It's this same principle that led people to suppose, once upon a time, that the world was flat. On our small personal scale, the Earth, like the cup, the doughnut, the pumpkin, and the bottle to the ant, looks flat to us. A manifold, in general though vague terms, is a topological space that shares this property. On a small enough scale, it looks "flat" i.e. it behaves like Euclidean space.

More precisely, an m-manifold is a very well-behaved class of topological space, which is Hausdorff, second countable, and locally m-Euclidean. That is, for every point x of our manifold X, there exists a neighborhood of x which is homeomorphic to an open set in $\Re^m$. The most obvious example of a manifold is $\Re^m$ itself, of course. Spheres, like our pumpkin, are another example: S^2 is a 2-manifold. So are tori, like the doughnut, or in fact the coffee cup. The Klein bottle is another 2-manifold. For any n, a more abstract example of a manifold is the matrix group $SL(n,\Re)$, which is an (n^2-1)-manifold.

Manifolds inherit a great many desirable topological properties from Euclidean space. In addition to being Hausdorff and second countable by definition, they are locally compact, locally path-connected, and regular. By the Urysohn metrization theorem, they are also metrizable. Munkres also shows with his Theorems 32.2 and 41.4 that manifolds are normal and paracompact. A space is paracompact if given any open covering of the space, one can find a refinement of the covering such that every point in the space lies in only finitely many elements of the refinement. Manifolds need not be compact, but they're the next best things.

It should come as very little surprise, given these properties, that any m-manifold can be embedded into $\Re^N$ for some finite *N*. In fact, the necessary value of *N* is at most $2m+1$. However, proving this and proving that a general manifold is embeddable are tricky. The special case of compact manifolds, with some unspecified *N*, is considerably easier and uses the idea of a partition of unity. Given a finite collection of open sets $U_1, U_2, \ldots, U_N$ in a space *X*, a partition of unity dominated by $\{U_i\}$ is a family of functions $\phi_i : X \to [0,1]$ such that each ϕ_i is 0 outside of U_i, and, at every point *x* of *X*, $\sum_{i=1}^{N} \phi_i(x) = 1$. Now suppose *X* is a compact manifold. Then we can take as our $\{U_i\}$ a finite open cover for *X*, for which each U_i is homeomorphic to an open set of $\Re^m$. Then there exist embeddings $g_i : U_i \to \Re^m$ for each *i*. Let $\phi_1, \phi_2, \ldots, \phi_N$ be a partition of unity dominated by the U_i. Define $h_i(x) = g_i(x)\phi_i(x)$ on U_i, and $(0,0,\ldots,0)$ on the complement of the support of ϕ_i. Recall that the support of a function $\gamma : X \to \Re$ is the closure of the set $\gamma^{-1}(\Re-\{0\})$. Then the function $F(x)=(\phi_1(x),\ldots,\phi_N(x),h_1(x),\ldots h_2(x))$ is an embedding of X in the outrageously large space $\Re^{n(m+1)}$.

To improve our embedding theorem and squeeze a compact manifold into a potentially much smaller Euclidean space, it is helpful to introduce the idea of topological dimension. There are several possible ways to define the dimension of a topological space. Many agree with our intuitive algebraic understanding of dimension, which tells us that $\Re^m$ ought to have dimension m. One of the more intuitive definitions is due to Menger and Urysohn. It proceeds recursively as follows: First, the empty set and only the empty set is declared to have dimension –1. Then for any other topological space *X*, the dimension of *X*, denoted $\dim X$, has $\dim X \le n$ if there is a basis for its topology, B, such that, for every $B \in \mathrm{B}$, $\text{bndy } B \le n-1$. Recall that $\text{bndy } A$, the boundary of a set *A*, is defined by $\text{bndy } A = \overline{A} \cap \overline{(X-A)}$. Conventionally, $\dim X$ is said to equal the least *n* for which this holds.

For example, a discrete space has dimension 0, as we can use as a basis all the one-point subsets, which have empty boundary. Likewise, the set of irrational numbers is also 0-dimensional: any neighborhood of an irrational point *p* contains an interval with rational endpoints, which has no boundary. Also, $\Re$, which has as a basis the open intervals whose boundaries are discrete pairs of points, can have dimension no greater than 1, and in fact $\dim \Re = 1$, just as expected. The proof that $\dim \Re^n \le n$ is a matter of induction, whereas the proof that $\dim \Re^n = n$ exactly is decidedly nontrivial.

One can go on to prove that, in separable metrizable spaces, a subspace cannot have higher dimension than its mother space, and thence that compact manifolds, which can be expressed as finite unions of open sets which have dimension no more than *m*, have dimension no more than *m*. Using some function space theory, Hurewicz and Wallman prove as their Theorem V2 that any compact *m*-manifold can be embedded in $\Re^{2m+1}$.

It is possible to prove a more general embedding theorem that does not rely on metrizability but on local compactness. Thus, this embedding theorem is still perfectly applicable to manifolds. Munkres presents this as Exercise 6 of §50, and it utilizes the Lebesgue definition of topological dimension. Although less intuitive than Menger's and Urysohn's, Lebesgue's definition of topological dimension is standard. In addition, since the Lebesgue and the Menger and Urysohn ideas of dimension agree on compact spaces, all of our previous results from Hurewicz and Wallman still hold. Lebesgue's notion of dimension is as follows: First, a collection of subsets of a space X is said to have order $m+1$ if there is a point of X that lies in $m+1$ elements of the collection, but no point lies in more than $m+1$. A space X is then said to be finite-dimensional if there is some integer m such that any open covering of X can be refined into an open covering with order at most $m+1$. The least integer m for which this holds is then the dimension of X.

Again, a discrete space has dimension 0 since any open covering can be refined into the collection of one-point subsets; every point lies in one and only one element of this covering, so the dimension of the space is at most 0, hence exactly 0. Also, given any open covering of the real line, we may easily construct a refinement such that no more than two covering sets intersect at any given point, and thus $\Re$ can have dimension no more than 1.

Now, consider any compact subset A of a general *m*-manifold X. A can be covered by finitely many open sets $U_1, U_2, \ldots, U_k$ homeomorphic to open sets in $\Re^m$. A

is a union of closed subsets homeomorphic to closed subsets of $\Re^m$, which must have dimension less than or equal to m, and therefore A itself has dimension less than or equal to m. This segues nicely into a theorem given as an exercise in Munkres that states: any locally compact second countable Hausdorff space, such that every compact subspace has dimension less than or equal to m, is homeomorphic to a closed subset of $\Re^{2m+1}$. The proof of this relies on some of the theory of function spaces, similar in some respects to the proof for compact spaces given in Hurewicz and Wallman. The basic idea is to consider $C\left(X, \Re^{2m+1}\right)$ under the uniform metric, and show that it has a dense subset with certain properties. The interested party may pursue this further. It follows soon thereafter that an m-manifold cannot have dimension greater than m. Though it has dimension precisely m, as one would expect, actually proving this definite result is a difficult task involving algebraic topology well beyond the scope of this brief document.

Though this theorem yields an upper bound on how many dimensions a Euclidean space must have to contain a homeomorphic copy of a given manifold, it does not offer a lower bound. Consider 2-manifolds, for example. The sphere and the torus—our pumpkin and doughnut—both exist happily as subspaces of $\Re^3$, as opposed to the suggested $\Re^5$. Our one-sided Klein bottle, also a 2-manifold, can be embedded in $\Re^4$. The projective plane, homeomorphic to the space of all lines through the origin in $\Re^3$ and to the quotient space obtained from S^2 by identifying pairs of antipodal points is also a 2-manifold that can be embedded in $\Re^4$. (Though it did not, alas, appear at our hypothetical picnic.) The $\left(n^2 - 1\right)$-manifold $SL(n, \Re)$ obtains its topology when considered as a space of n^2-vectors, making it a subspace of $\Re^{n^2}$. However, it can be trivially embedded in a Euclidean space with dimension only one greater than its own. $\Re^m$, the prototypical m-manifold, requires no embedding at all.

The embedding theorems mentioned here show that any m-manifold can be embedded in a finite-dimensional Euclidean space, while they do not by any means suggest the best possible embedding. The last embedding theorem can not in general be refined any further, since there are m-manifolds, though rare and exotic, which cannot be embedded in $\Re^N$ for any $N < 2m+1$.

WORKS CITED

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