Recollection Sheet

- χ is a character on $\mathcal{O}_K$, the ring of integers of a quadratic imaginary field K of discriminant $-D$.
- $f = \sum a_m q^m$ is a weight 2 newform of level N, a rational prime inert in K.
- $u = \#(\mathcal{O}_K^\times/\mathbb{Z}^\times)$ and $\epsilon(p) =$ Legendre symbol $\left(\frac{-D}{p}\right)$.
- R_i is a maximal order in $\mathcal{B}$, the quaternion algebra ramifying at N and ∞.
- $\Gamma_i = R_i^\times/\mathbb{Z}^\times$ and $\omega_i = \#\Gamma_i =$ half the units of R_i.
- $\{e_i\}$ is the natural basis for $\mathrm{Pic}(X_N)$ where we think of $X_N = \coprod X_i$.
- $\Theta = \sum_m t_m q^m \in \mathrm{End}(\mathrm{Pic}(X_n))[[q]]$.
- $\Phi(e, e') = \langle \Theta e, e' \rangle$.
- x_A is the image of $x \in X_N$ after being acted on by $A \in \mathrm{Pic}(\mathcal{O}_K)$.
- $c(\chi) = \sum \chi(A)^{-1} x_A$.
- $\theta_A(z) = \frac{1}{2u} \sum\limits_{\lambda \in \mathfrak{a}} q^{\mathbb{N}\lambda/\mathbb{N}\mathfrak{a}} = \frac{1}{2u} \sum\limits_{m \geq 0} r_A(m) q^m$

Heights and the Special Values of L-series

B.H. Gross

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Introduction

Let:

- $f = \sum_{m=1}^{\infty} a_m q^m$ is a cusp form of weight 2 on $\Gamma_0(N)$, with N prime.

- $L(f, s) = \sum_{m=1}^{\infty} a_m m^{-s}$ the associated Hecke L-series given by the Mellin transform.

and

- K is a quadratic imaginary field with discriminant $-D$ and ring of integers $\mathcal{O}_K$.

- $\chi : \mathrm{Pic}(\mathcal{O}_K) \to \mathbb{C}^\times$ where $\mathrm{Pic}(\mathcal{O}_K)$ is the ideal class group of K.

- $L(\chi, s) = \sum_{\mathfrak{a} \subset \mathcal{O}_K} \chi(\mathfrak{a}) \, (\mathbb{N}\mathfrak{a})^{-s} = \sum_{m=1}^{\infty} b_m m^{-s}$.

When the Rankin-Selberg product $L(f, \chi, s)$ has an odd functional equation, the famous Gross-Zagier paper gives an explicit formula for $L'(f, \chi, 1)$ in terms of heights.

When the functional equation is even, Gross gives an explicit formula for $L(f, \chi, 1)$ in terms of heights.

Introduction, cont.

Rankin-Selberg: The combined L-series for f and χ is

$$L(f, \chi, s) = \zeta(2s-2) \sum_{m=1}^{\infty} a_m b_m m^{-s}$$

For a slightly modified function, $L^*(f, \chi, s)$, Rankin-Selberg also gives a functional equation

$$L^*(f, \chi, 2-s) = -\epsilon(N) L^*(f, \chi, s).$$

where $\epsilon(N)$ is the Legendre symbol $\left(\frac{-D}{N}\right)$.

Heights

- Gross-Zagier: Neron-Tate Heights of Heegner points on a modular curve.
- Gross: "Heights" of special points on a curve X_N associated to a definite quaternion algebra $\mathcal{B}$ ramifying at N and ∞.

Outline

- We start with an L-series defined as the Rankin-Selberg product of $L(f, s)$ and $L(\chi, s)$.
- Then, given a prime N, we construct a curve, X_N, together with a height pairing and the Hecke module structure on $\mathrm{Pic}(X_N)$.
- We give points on this curve which are associated to the field K and a character χ.
- Finally, we compute the special value of the L-series as the height pairing on the f-component of these points.

Rankin-Selberg L-Series

- Given a cusp form $f = \sum_{m=1}^{\infty} a_m q^m$ we can consider the associated L-series given by the Mellin transform:

$$\begin{aligned} L(f,s) &= \sum_{m=1}^{\infty} a_m m^{-s} \\ &= \prod_{p \text{ prime}} [(1-\alpha_1(p)p^{-s})(1-\alpha_2(p)p^{-s})]^{-1} \end{aligned}$$

- We'll take f to be level N, a prime.

- Let K be an imaginary quadratic field with ring of integers $\mathcal{O}_K$. Let χ be a character on the ideal class group $\mathrm{Pic}(\mathcal{O}_K)$. Then there is an associated L-series:

$$\begin{aligned} L(\chi,s) &= \sum_{\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_K)} \chi(\mathfrak{a})(\mathbb{N}\mathfrak{a})^{-s} \\ &= \prod_{p \text{ prime}} [(1-\beta_1(p)p^{-s})(1-\beta_2(p)p^{-s})]^{-1} \end{aligned}$$

- Rankin and Selberg provide a natural way to combine these to give:

$$L(f,\chi,s) = \prod_{p \text{ prime}} \left(\prod_{i=1,2} \prod_{j=1,2} (1-\alpha_i(p)\beta_j(p)p^{-s}) \right)^{-1}$$

Properties from Rankin-Selberg

The Rankin-Selberg method gives two important properties of $L(f,\chi,s)$:

1. $L^*(f,\chi,s) := (2\pi)^{-2s}\Gamma(s)^2(ND)^s L(f,\chi,s)$ has analytic continuation to the whole plane, where $D = -\mathrm{Disc}(K)$.

2. $L^*(f,\chi,2-s) = -\epsilon(N)L^*(f,\chi,s)$.

- Note: The critical value is $s = 1$.

- Gross and Zagier examine in detail the case of an odd functional equation, i.e. $\epsilon(N) = 1$, i.e. N split in K. Hence, $L(f,\chi,1) = 0$ and the first derivative is the interesting value.

- On the other hand, Gross considers the case of an even functional equation, i.e. $\epsilon(N) = -1$, i.e. N inert in K. Here the value at $s = 1$ is pertinent.

Main Theorem

Theorem. *Let f be a weight 2 newform on $\Gamma_0(N)$ with N an inert prime in an imaginary quadratic field K with discriminant $-D$. Let χ be a character of $Pic(\mathcal{O}_K)$. Then*

$$L(f,\chi,1) = \frac{1}{u^2\sqrt{D}}\langle f, f\rangle_{Pet}\langle c_f(\chi), c_f(\chi)\rangle$$

where $u = \#(\mathcal{O}_K^\times/\mathbb{Z}^\times)$, $\langle\cdot,\cdot\rangle_{Pet}$ is the Petersson inner product, and $\langle c_f(\chi), c_f(\chi)\rangle$ is the height pairing of a certain divisor, $c_f(\chi)$, with itself on a certain curve, X_N.

Construction of X_N

- Let $\mathcal{B}$ the unique quaternion algebra over $\mathbb{Q}$ ramifying at N and ∞, i.e. $\mathcal{B}\otimes_{\mathbb{Q}}\mathbb{Q}_N$ and $\mathcal{B}\otimes_{\mathbb{Q}}\mathbb{R}$ are division algebras over $\mathbb{Q}_N$ and $\mathbb{R}$, resp., and for all other primes p, $\mathcal{B} \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong M_2(\mathbb{Q}_p)$.

- Fix a maximal order $R \subset \mathcal{B}$

- *Equivalence of Left Ideals of* R: $I \sim J \Leftrightarrow \exists b \in \mathcal{B}^\times$ such that $J = Ib$ (where $\mathcal{B}^\times$ are the units of $\mathcal{B}$).

- Let $\{I_1 = R, I_2, ..., I_n\}$ represent the ideal classes. n is called the class number of $\mathcal{B}$ and is independent of R.

- For every I_i the associated maximal order is

$$R_i = \{b \in \mathcal{B} | I_i b \subset I_i\}$$

- Set $\Gamma_i = R_i^\times/\mathbb{Z}^\times$ and $\omega_i = \#\Gamma_i =$ half the number of units of R_i.

Adeles on $\mathcal{B}$

- As usual $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ and $\widehat{\mathbb{Q}} = \widehat{\mathbb{Z}} \otimes \mathbb{Q}$.
- For every prime p, we can define the local components of the maximal orders $R_{i,p} = R_i \otimes \mathbb{Z}_p$ and then set $\widehat{R_i} = R_i \otimes \widehat{\mathbb{Z}}$.
- Likewise, $\widehat{\mathcal{B}} = \mathcal{B} \otimes \widehat{\mathbb{Q}}$.
- Every ideal I of R is locally principal, so $\exists g_{I,p} \in R_p^\times \backslash \mathcal{B}_p^\times$ such that $I_p = R_p g_{I,p}$.
- Then there's a bijection

$$\begin{aligned} \{\text{ideals of } R\} &\longleftrightarrow \widehat{R}^\times \backslash \widehat{\mathcal{B}}^\times \\ I &\longleftrightarrow g_I = (\ldots g_{I,p} \ldots) \end{aligned}$$

- Recall the equivalence relation between left ideals and get

$$n = \# \widehat{R}^\times \backslash \widehat{\mathcal{B}}^\times / \mathcal{B}^\times$$

- So choose representatives g_i such that

$$\widehat{\mathcal{B}}^\times = \bigcup_{i=1}^{n} \widehat{R}^\times g_i \mathcal{B}^\times$$

- Can recover R_i from g_i by

$$R_i = \mathcal{B} \cap g_i^{-1} \widehat{R} g_i$$

The Curve X_N

- Every quaternion algebra, $\mathcal{C}$ over $\mathbb{Q}$, has an associated genus 0 curve $Y_{\mathcal{C}}$ where for any $\mathbb{Q}$-algebra, W, the W points of $Y_{\mathcal{C}}$ are given by

$$Y_{\mathcal{C}}(W) = \{\alpha \in \mathcal{C} \otimes_{\mathbb{Q}} W | \alpha \neq 0, \mathrm{Tr}\alpha = \mathbb{N}\alpha = 0\}/W^{\times}$$

- Let $Y = Y_{\mathcal{B}}$ be the curve associated to $\mathcal{B}$. Notice $\mathcal{B}^{\times}$ acts on (the right of) Y by conjugation.
- Then set

$$X_N = (\widehat{R}^{\times} \backslash \widehat{\mathcal{B}}^{\times} \times Y)/\mathcal{B}^{\times}$$

- Then it is more natural to think of X_N as

$$\begin{aligned} X_N &= \coprod_{i=1}^{n} Y/\Gamma_i \\ (\widehat{R}^{\times} g_i, y) \bmod \mathcal{B}^{\times} &\mapsto y \bmod \Gamma_i \text{ on } X_i := Y/\Gamma_i \end{aligned}$$

- Each $X_i \cong \mathbb{P}^1/\Gamma_i$.

$\mathbf{Pic}(X_N)$ and The Height Pairing

- Definitions of $\mathrm{Pic}(X_N)$.

  1. It is the group of line bundles on X_N.
  2. It is the group of invertible sheafs of X_N.
  3. When X_N is nice, it is isomorphic to the divisor class group.

- $\mathrm{Pic}(X_N) = \mathrm{Pic}(\coprod_{i=1}^n X_i) = \coprod_{i=1}^n \mathrm{Pic}(X_i)$
- Each $X_i \cong \mathbb{P}^1$. And $\mathrm{Pic}(\mathbb{P}^1) \cong \mathbb{Z}$.
- Hence, $\mathrm{Pic}(X_N) \cong \mathbb{Z}^n$.
- So let e_i be the generator of $\mathrm{Pic}(X_i)$. Hence $\{e_i\}$ is a basis for $\mathrm{Pic}(X_N)$.
- Define the height pairing on basis elements:

$$\langle e_i, e_j \rangle = \begin{cases} \omega_i & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

  and extend biadditively to all of $\mathrm{Pic}(X_N)$.

- $\langle \cdot, \cdot \rangle$ gives an embedding

$$\mathrm{Pic}(X_N)^\vee = \mathrm{Hom}(\mathrm{Pic}(X_N), \mathbb{Z}) \hookrightarrow \mathrm{Pic}(X_N) \otimes \mathbb{Q}$$

$$e_i^\vee \mapsto e_i/\omega_i$$

Relationship Between X_N and Supersingular Elliptic Curves

Recall that an elliptic curve $E/\mathbb{F}_q$ is call supersingular if $E(\overline{\mathbb{F}_q})$ has no points of order $\mathrm{char}(\mathbb{F}_q)$.

Lemma. *Let $\mathbb{F}$ be an algebraically closed field of characteristic N and n be the class number of $\mathcal{B}$. Then there are n distinct isomorphism classes of supersingular elliptic curves over $\mathbb{F}$ and furthermore*

$$End(E_i) \cong R_i$$

This sets up bijections:

$$\left\{ \begin{matrix} \text{Components} \\ \text{of } X_N \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{Maximal} \\ \text{Orders of } \mathcal{B} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{Supersingular} \\ \text{ECs over } \mathbb{F} \end{matrix} \right\}$$

$$X_i \longleftrightarrow R_i \longleftrightarrow E_i$$

Hecke Operators

- Want to use these associations to define a Hecke operator on $\text{Pic}(X_N)$.

$$t_m[E_i] := \sum_{\lambda_{ij}} [E_j]$$

where $\lambda_{ij} \in \text{Hom}^m(E_i, E_j)/\text{Aut}(E_j)$.

- Then use the bijections to define over the e_i instead of the E_i.

- **Adjointness:** For any two divisors, e and $e^\vee$,

$$\langle t_m e, e^\vee \rangle = \langle e, t_m e^\vee \rangle.$$

- Let $\text{Pic}_{\mathbb{Q}}(X_N) = \text{Pic}(X_N) \otimes \mathbb{Q}$. If we then consider $t_m \in \text{End}(\text{Pic}_{\mathbb{Q}}(X_N))$ with the $\{e_i\}$ basis, we get the Brandt matrices $t_m = B(m) \in M_n(\mathbb{Q})$ with entries

$$[B(m)]_{ij} = B_{ij}(m) = \#\ \text{Hom}^m(E_i, E_j)/\text{Aut}(E_j)$$

and set $B_{ij}(0) = \frac{1}{2\omega_j}$.

- In fact, $B(1) = I_n$ and $B(m) \in M_n(\mathbb{Z})$ for $m \geq 1$

The Generating Series Θ

Define

$$\Theta = \sum_{m \geq 0} t_m q^m \in \mathrm{End}(\mathrm{Pic}_{\mathbb{Q}}(X_N))[[q]]$$

Proposition. *Θ is a weight 2 modular form of level N, valued in End($Pic_{\mathbb{Q}}(X_N)$).*

Proof. Consider

$$\begin{aligned}[\Theta]_{ij} &= \sum_{m \geq 0} B_{ij}(m) q^m \\ &= \frac{1}{\#\mathrm{Aut}(E_j)} \sum_{\phi \in \mathrm{Hom}(E_i, E_j)} q^{\deg \phi}\end{aligned}$$

If we now recall that $\mathrm{End}(E_i) \cong R_i$ then we see that

$$\mathrm{Hom}(E_i, E_j) \cong I_j^{-1} I_i.$$

If $\phi \mapsto b$, then $\deg \phi = \frac{\mathbb{N}b}{\mathbb{N}(I_j^{-1} I_i)}$. So

$$[\Theta]_{ij} = \frac{1}{2\omega_j} \sum_{b \in I_j^{-1} I_i} q^{\frac{\mathbb{N}b}{\mathbb{N}(I_j^{-1} I_i)}}$$

This is just a standard θ-series over a lattice and hence a modular form of weight 2. □

Corollary. *For any e and $e^\vee$, $\Phi(e, e^\vee) := \langle \Theta e, e^\vee \rangle$ is a $\mathbb{C}$-valued weight 2 modular form over $\Gamma_0(N)$.*

Next Time

- We will take a closer look at the Hecke operators t_m and see how they relate to the usual Hecke operators on modular forms.
- Using this relation, we will be able to define and talk about Hecke eigencomponents of $\mathrm{Pic}(X_N)$ and how Φ acts on them.
- Then we will define a set of points on X_N given by the field K and examine their properties.
- Next, we'll fix a distinguished element of $\mathrm{Pic}(X_N)$ and show how it allows us to rephrase the main identity.
- Lastly, we'll go over the final proof of the main identity with as many details as time allows.

Last Time

- We had a Rankin-Selberg L-series $L(f, \chi, s)$ determined by
  1. A character χ of $\mathrm{Pic}(\mathcal{O}_K)$ where K is a quadratic imaginary field.
  2. A cusp form f of level N, a prime inert in K.

  And it had an even functional equation.

Theorem.
$$L(f, \chi, 1) = \frac{1}{u^2\sqrt{D}}\langle f, f\rangle_{Pet}\langle c_f(\chi), c_f(\chi)\rangle$$
where $-D = disc(K)$, $u = \#(\mathcal{O}_K^\times/\mathbb{Z}^\times)$, $\langle\cdot,\cdot\rangle_{Pet}$ *is the Petersson inner product, and* $c_f(\chi)$ *is an element of* $Pic(X_N)$.

- We then used the unique quaternion algebra $\mathcal{B}$ that ramified at N and ∞ to construct the curve
$$X_N = \coprod_{i=1}^{n} Y/\Gamma_i$$
  with $Y \cong \mathbb{P}^1$.

- Next, we defined a "height" pairing $\langle\cdot,\cdot\rangle$ on $\mathrm{Pic}(X_N)$ and Hecke operators t_m based on the relations between X_N and supersingular elliptic curves.

- Lastly, we had modular forms Θ valued in $\mathrm{End}(\mathrm{Pic}_{\mathbb{Q}}(X_N))$ and Φ valued in $\mathbb{C}$.

Hecke Structure

- Let $\mathcal{M}_{\mathbb{Q}}$ denote the modular forms of weight 2 on $\Gamma_0(N)$ with rational coefficients, and let T_m be the standard Hecke operators on $\mathcal{M}_{\mathbb{Q}}$. Then $\mathbb{T} = \mathbb{Q}[...T_m...]$ is a subalgebra of $\text{End}(\mathcal{M}_{\mathbb{Q}})$.
- Similarly $\mathbb{B} = \mathbb{Q}[...t_m...]$ is a subalgebra of $\text{End}(\text{Pic}_{\mathbb{Q}}(X_N))$.
- Recall $B(m)$ was the matrix for t_m in the $\{e_i\}$ basis.
- By establishing $\text{Trace}(T_m) = \text{Trace}(B(m))$, Eichler showed that there is a ring isomorphism $\mathbb{T} \cong \mathbb{B}$ that takes $T_m \mapsto t_m$. This leads to the following proposition and its corollary:
- Recall, we defined a generating series

$$\Theta = \sum_{m \geq 0} t_m q^m \in \text{End}(\text{Pic}_{\mathbb{Q}}(X_N))[[q]]$$

and showed that it was a weight 2 modular form and that for any e and $e^\vee$, $\Phi(e, e^\vee) := \langle \Theta e, e^\vee \rangle$ is a $\mathbb{C}$-valued weight 2 modular form over $\Gamma_0(N)$.

Proposition. $\Theta | T_m = t_m \circ \Theta = \Theta \circ t_m$

Corollary. $\Phi(e, e^\vee) | T_m = \Phi(t_m e, e^\vee) = \Phi(e, t_m e^\vee)$

Hecke Eigencomponents

Modular Forms

- $\mathcal{M}_{\mathbb{Q}}$ is a free $\mathbb{T}$-module of rank 1 by the multiplicity one theorem since every eigenform in $\mathcal{M}_{\mathbb{R}}$ is a newform of level N. (Since there are no weight 2 oldforms and N is prime.)
- $f = \sum a_m q^m$ is an eigenfunction of $\mathbb{T}$, so $f|T_m = a_m f$ for each m. Set

$$\mathcal{M}_f = \bigcap_m \ker(T_m - a_m)$$

- $\mathcal{M}_f = \mathbb{C}f \subset \mathcal{M}_{\mathbb{C}}$ which is an n-dimensional vector space.

Pic(X_N)

- $\mathrm{Pic}_{\mathbb{Q}}(X_N)$ is a free $\mathbb{B}$ module of rank 1.
- Since $\mathbb{T} \cong \mathbb{B}$, we get $\mathcal{M}_{\mathbb{C}} \cong \mathbb{T}_{\mathbb{C}} \cong \mathbb{B}_{\mathbb{C}} \cong \mathrm{Pic}_{\mathbb{C}}(X_N)$.
- Define $\mathrm{Pic}_{\mathbb{C}}(X_N)_f$ to be the corresponding 1-dimensional subspace of $\mathrm{Pic}_{\mathbb{C}}(X_N)$ and for a divisor e, let e_f to be its projection onto this f-eigencomponent.

$$\begin{array}{ccc} \mathcal{M}_{\mathbb{C}} & & \mathrm{Pic}_{\mathbb{C}}(X_N) \\ \downarrow & & \downarrow \\ \mathcal{M}_f & & \mathrm{Pic}_{\mathbb{C}}(X_N)_f \end{array}$$

Tautology: For any divisor c, $t_m c_f = a_m c_f$.

Φ on the f-eigencomponent

- Recall $\Phi : \mathrm{Pic}(X_N) \times \mathrm{Pic}(X_N) \to \mathcal{M}$ defined by

$$\Phi(c, c') = \langle \Theta c, c' \rangle$$

- Consider

$$\begin{aligned} \Phi(c_f, c'_f) &= \langle \Theta c_f, c'_f \rangle \\ &= \left\langle \left(\sum t_m q^m \right) c_f, c'_f \right\rangle \\ &= \left\langle \left(\sum a_m q^m \right) c_f, c'_f \right\rangle \\ &= f \cdot \langle c_f, c'_f \rangle \end{aligned}$$

Therefore $\langle f, \Phi(c_f, c'_f) \rangle_{\mathrm{Pet}} = \langle f, f \rangle_{\mathrm{Pet}} \cdot \langle c_f, c'_f \rangle$.

- Since $\Phi(t_m c, c') = \Phi(c, c') | T_m$ and $\langle c_f, c' \rangle = \langle c_f, c'_f \rangle$, then

$$\begin{aligned} \Phi(c_f, c'_f) &= \Phi(c, c')_f \\ \Longrightarrow \langle f, \Phi(c, c') \rangle_{\mathrm{Pet}} &= \langle f, \Phi(c_f, c'_f) \rangle_{\mathrm{Pet}} \\ &= \langle f, f \rangle_{\mathrm{Pet}} \cdot \langle c_f, c'_f \rangle \end{aligned}$$

$$\boxed{\langle f, \Phi(c, c') \rangle_{\mathrm{Pet}} = \langle f, f \rangle_{\mathrm{Pet}} \cdot \langle c_f, c'_f \rangle}$$

Special Points and Fine Structure of X_N

- Recall
$$X_N = \coprod_{i=1}^{n} Y/\Gamma_i = (\widehat{R}^\times \backslash \widehat{\mathcal{B}}^\times \times Y)/\mathcal{B}^\times$$
$$Y(K) = \{\alpha \in \mathcal{B} \otimes K | \alpha \neq 0, \mathrm{Tr}\alpha = \mathbb{N}\alpha = 0\}/K^\times.$$

- Let the special points be defined as
$$X_N(K) := (\widehat{R}^\times \backslash \widehat{\mathcal{B}}^\times \times Y(K))/\mathcal{B}^\times$$

- There is an identification $\mathrm{Hom}(K, \mathcal{B}) = Y(K)$ by

  $\rho \mapsto \alpha \in Y(K)$ such that $\forall k \in K^\times$, $\rho(k)^{-1}\alpha\rho(k) = \alpha\rho\left(k/\bar{k}\right)$

- Under this identification, the $\mathcal{B}^\times$ action becomes conjugation, $(\rho \cdot b)(k) := b^{-1}\rho(k)b$.

- Let $x = (h, y_\rho) \bmod \mathcal{B}^\times \in X_N(K)$, where $\rho : K \hookrightarrow \mathcal{B}$. We then define the discriminant as $\mathrm{disc}(x) := \mathrm{disc}(\mathcal{O})$ where $\rho(K) \cap h^{-1}\widehat{R}h = \rho(\mathcal{O})$ for some order $\mathcal{O} \subset K$.

- Let $X_N(K, \mathcal{O}) = \{x \in X_N(K) | \mathrm{disc}(x) = \mathrm{disc}(\mathcal{O})\}$. Then we can make the following decomposition:
$$X_N(K) = \bigcup_{\mathcal{O} \subset K} X_N(K, \mathcal{O})$$

- We also get a (non-obvious) transitive action of $\mathrm{Pic}(\mathcal{O})$ on $X_N(K, \mathcal{O})$ which we will denote by $x \mapsto x_A$.

The Distinguished Divisor

- Recall the main theorem we are trying to prove is

$$L(f,\chi,1) = \frac{1}{u^2\sqrt{D}}\langle f,f\rangle_{\mathrm{Pet}}\langle c_f(\chi), c_f(\chi)\rangle.$$

- Fix a special point $x \in X_N$ of discriminant D. Define

$$c(\chi) = \sum_{A\in \mathrm{Pic}(\mathcal{O}_K)} \chi(A)^{-1} x_A$$

  and let $c_f(\chi)$ be its f-component.

- From what we've shown previously then

$$\langle f,f\rangle_{\mathrm{Pet}}\langle c_f(\chi), c_f(\chi)\rangle = \langle f, \Phi(c(\chi), c(\chi))\rangle_{\mathrm{Pet}}$$

- So the identity to be shown is

$$L(f,\chi,1) = \frac{1}{u^2\sqrt{D}}\langle f, \Phi(c(\chi), c(\chi))\rangle_{\mathrm{Pet}}.$$

- Rankin and Selberg, through analytic methods, also give us $L(f,\chi,1)$ as a Petersson inner product.

Eliminating χ Dependence

- Putting in our distinguished divisor $c(\chi)$ yields

$$\begin{aligned}\langle f, \Phi(c(\chi), c(\chi))\rangle_{\text{Pet}} &= \left\langle f, \sum_{A,B\in\text{Pic}(\mathcal{O}_K)} \chi(A)^{-1}\chi(B)\Phi(x_A, x_B)\right\rangle_{\text{Pet}} \\ &= \left\langle f, \sum_A \chi(A) \sum_B \Phi(x_B, x_{AB})\right\rangle_{\text{Pet}} \\ &= \sum_A \chi(A) \left\langle f, \sum_B \Phi(x_B, x_{AB})\right\rangle_{\text{Pet}}\end{aligned}$$

- Likewise,

$$L(f, \chi, s) = \sum_A \chi(A) L(f, A, s)$$

where

$$L(f, A, s) = \left[\sum_{\substack{(m,N)=1 \\ m\geq 1}} \frac{\epsilon(m)}{m^{2s-1}}\right] \left[\sum_{m\geq 1} \frac{a_m r_A(m)}{m^s}\right]$$

$\epsilon(m) = \left(\frac{-D}{m}\right)$ and the $r_A(m)$ are given by fixing an ideal $\mathfrak{a}$ in the ideal class A and setting

$$\theta_A(z) = \frac{1}{2u} \sum_{\lambda\in\mathfrak{a}} q^{\mathbb{N}\lambda/\mathbb{N}\mathfrak{a}} = \frac{1}{2u} \sum_{m\geq 0} r_A(m) q^m$$

- So the main theorem is now equivalent to showing

$$L(f, A, 1) = \frac{1}{u^2\sqrt{D}} \left\langle f, \sum_B \Phi(x_B, x_{AB})\right\rangle_{\text{Pet}}$$

Method to Prove Main Theoreom

1. Use Rankin-Selberg to obtain $L(f, A, 1)$ as a Petersson inner product of f with a modular form constructed from an Eisenstein series. Then perform a trace computation to compute the Fourier coefficients of this form.

2. Compute Fourier coefficients of $\sum_B \Phi(x_B, x_{AB})$ using algebraic/geometric properties.

3. Compare.

Rankin-Selberg Method

- Define a new Eisenstein series

$$E_{ND}(s,z) = \sum_{\substack{(m,N)=1 \\ m\geq 1}} \frac{\epsilon(m)}{m^{2s-1}} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(ND)} \frac{\epsilon(d)}{(cz+d)} \frac{y^s}{|cz+d|^{2s}}$$

- $E_{ND}(s,z)$ is a weight 1 modular form of level ND with character ϵ.

- Then standard integration techniques developed by Rankin and Selberg give

$$(4\pi)^{-s}\Gamma(s)L(f,A,s) = \iint_{\mathcal{F}_{ND}} f(z)\overline{\theta_A(z)E_{ND}(\bar{s}-1,z)}dxdy$$

- Therefore at $s=1$ we get

$$\frac{1}{4\pi}L(f,A,1) = \langle f, \theta_A E_{ND}(0,z)\rangle_{\text{Pet}}$$

- Now the main theorem can be further reduced to showing

$$\langle f, \theta_A E_{ND}(0,z)\rangle_{\text{Pet}} = \frac{4\pi}{u^2\sqrt{D}}\left\langle f, \sum_B \Phi(x_B, x_{AB})\right\rangle_{\text{Pet}}$$

- We can do so by showing the stronger statement

$$\theta_A E_{ND}(0,z) = \frac{4\pi}{u^2\sqrt{D}}\sum_B \Phi(x_B, x_{AB})$$

(for a specific $x \in X_N$) by computing Fourier coefficients of each.

The Fourier Coefficients of $\sum_B \Phi(x_B, x_{AB})$

$$\begin{aligned} \Phi(x_B, x_{AB}) &= \langle \Theta x_B, x_{AB} \rangle \\ &= \sum_m \langle t_m x_B, x_{AB} \rangle q^m \end{aligned}$$

So we need to compute $\langle t_m x_B, x_{AB} \rangle = \langle x_B, t_m x_{AB} \rangle$.

- Recall we had the associations $X_i \longleftrightarrow R_i \longleftrightarrow E_i$.
- Then $\langle x_B, t_m x_{AB} \rangle = \frac{1}{2} \# \mathrm{Hom}^m(E_B, E_{AB})$ where E_B denotes the supersingular elliptic curve corresponding to x_B.
- Consider $\mathcal{B} = K + K\eta$ where $\eta^2 = -N$ and $\eta\alpha = \bar{\alpha}\eta$ $\forall \alpha \in K$. Let $\mathcal{D} = \left(\sqrt{-D}\right)$ be the different of $\mathcal{O}_K$. Let $\varepsilon^2 \equiv -N \mod D$.
- A theorem proved in a paper by one of Gross's students then states we can chose $x \in X_N$ such that

  $$\mathrm{End}(E_x) = \left\{ \alpha + \beta\eta | \alpha, \beta \in \mathcal{D}^{-1}, \alpha \equiv \varepsilon\beta \bmod \mathcal{O}_{\mathcal{D}} \right\}$$

  where $\mathcal{O}_{\mathcal{D}}$ is $\mathcal{O}_K$ localized at the prime $\mathcal{D}$.

  Note. *x may actually be chosen as an arbitrary special point on X_N with discriminant D. This changes $\Phi(x_B, x_{AB})$ by an oldform. Since we're taking the Petersson inner product with f, a newform, this doesn't affect the overall calculation.*

The Fourier Coefficients of $\sum_B \Phi(x_B, x_{AB})$, cont.

- Fix $\mathfrak{a}$ and $\mathfrak{b}$ ideals in the classes of A and B with $\mathfrak{a}$ and $\mathfrak{b}$ relatively prime to $\mathcal{D}$. Then the above lets us form a bijection between $\mathrm{Hom}(E_B, E_{AB})$ and the set

$$\left\{\alpha + \beta\eta | \alpha \in \mathcal{D}^{-1}\mathfrak{a}, \beta \in \mathcal{D}^{-1}\mathfrak{b}^{-1}\bar{\mathfrak{b}}\bar{\mathfrak{a}}, \alpha \equiv \varepsilon\beta \bmod \mathcal{O}_{\mathcal{D}}\right\}$$

  such that $\deg(\phi) \longleftrightarrow (\mathbb{N}\alpha + N\mathbb{N}\beta)/\mathbb{N}\mathfrak{a}$.

- We want to count the number of solutions to $\mathbb{N}\alpha + N\mathbb{N}\beta = m\mathbb{N}\mathfrak{a}$ with α, β as above.

- Recall $r_A(m) = \#\left\{\lambda \in \mathfrak{a} \text{ in the class } A | \mathbb{N}\lambda = m\mathbb{N}\mathfrak{a}\right\}$.

- Set $\delta(n) = \#$ of primes dividing both n and D. Then our Fourier coefficients are

$$\left\langle x_B, t_m x_{AB}\right\rangle = u^2 \sum_{n=0}^{mD/N} r_{A^{-1}}(mD - nN) 2^{\delta(n)} r_{AB^2}(n)$$

- These coefficients are the same as those arrived at through the Rankin-Selberg method up to the outside constant $\frac{4\pi}{u^2\sqrt{D}}$.

- Hence,

$$\left\langle f, \theta_A E_{ND}(0,z)\right\rangle_{\mathrm{Pet}} = \frac{4\pi}{u^2\sqrt{D}} \left\langle f, \sum_B \Phi(x_B, x_{AB})\right\rangle_{\mathrm{Pet}}$$

  and now the main theorem follows.