A Survey of the Dynamical Approaches to the 3*n*+1 Problem

by Eric Errthum

§1 Introduction and Set-Up

Originally a problem in graph theory, the $3n+1$ problem has been around since before the 1950s when it was first proposed by Lothar Collatz [3]. To state the conjecture, we first define the function $F:\mathbb{Z}^+\to\mathbb{Z}^+$ by:

$$F(x)=\begin{cases}3x+1 & x \text{ odd}\\ x/2 & x \text{ even}\end{cases}. \tag{1}$$

The conjecture states that for every $x\in\mathbb{Z}^+$ there exists an n such that $F^n(x)=\overbrace{F(F(\cdots(x)))}^{n\text{ times}}=1$. As an example, consider $x=3$. Then the orbit of x becomes

$$3\to10\to5\to16\to8\to4\to2\to1.$$

One readily sees that once 1 is reached, further iteration of F produces what is called the trivial cycle $1\to4\to2\to1$. The conjecture is known by many names including the Collatz conjecture, Ulam's problem, the Syracuse problem and many more [3]. In this paper, it shall simply be referred to as the *main conjecture*. Also, since F always takes an odd integer to an even integer, a further simplification can be made by defining the function $T:\mathbb{Z}^+\to\mathbb{Z}^+$ by:

$$T(x)=\begin{cases}\frac{3}{2}x+\frac{1}{2} & x \text{ odd}\\ \frac{1}{2}x & x \text{ even}\end{cases}. \tag{2}$$

The statement of the main conjecture is essentially still the same, replacing F by T.

The two possible counterexamples to the main conjecture are the existence of either a nontrivial cycle of positive integers or a divergent trajectory. Or more formally, the main conjecture is false if and only if *i)* there exists $n>2$ and $k\geq1$ such that $T^k(n)=n$, or if *ii)* there exists $n>2$ such that $\lim_{k\to\infty}T^k(n)=\infty$.

The iterative nature of the problem at hand suggests a dynamical approach should be taken. However, as most of dynamics deals with real-valued continuous functions, we need to embed the behavior of T into a continuous function defined on the reals. That is, we need to find a continuous $f:\mathbb{R}\to\mathbb{R}$ such that $f\big|_{\mathbb{Z}^+}=T$. Unfortunately, problems arise almost immediately.

Suppose we had such an *f*. Then for any positive integer *n* we know that $f(2n)=n$ and $f(2n+1)=3n+2>2n+1$. Therefore, as *f* is continuous, it takes the interval $[2n,2n+1]$ to the interval $[n,3n+2]$. A similar argument can now be applied to the interval $[3n+1,3n+2]$ and it becomes readily apparent that many divergent trajectories exist. In fact, for any positive integer *n*, the divergent trajectories in the interval $[n,n+1]$ form an uncountable Cantor set [2]. Nevertheless, there is still much to be gathered from the continuous translation.

There are, obviously, numerous functions that one can choose for *f*. However, there are some conditions on *f* that yield some canonical choices. First, since we will eventually want to apply the techniques of dynamics to *f*, it seems that *f* should be a C^∞ function. Although dynamics can theoretically handle piecewise differentiable functions, it's reasonable that in the effort of constructing the "nicest" form of *f*, piecewise differentiability would be undesirable. (Although, the conjecture is still open so perhaps this is a naive assumption.)

So now that *f* is differentiable, one must decide what its derivative should be at each integer. Two answers seem to arise. The first is to construct *f* such that for $x\in\mathbb{Z}^+$,

$$f'(x)=\begin{cases}\frac{3}{2} & x\text{ odd}\\ \frac{1}{2} & x\text{ even}\end{cases}. \tag{3}$$

In this way, *f* mimics the contraction or expansion properties of *T* in the neighborhood of an integer. The other option is to construct *f* so that

$$f'(x)=0\ \text{ for }\ x\in\mathbb{Z}^+. \tag{4}$$

The advantages of this option are that all integers are critical points of *f* and any orbit on the integers is super-attracting [4]. These two options shall now be considered.

§2 Functions That Act Like *T* Near Integers

We will first consider functions whose derivative satisfies Eq. 3. The most canonical choice for *f* is

$$f(x)=x+\frac{1}{4}-\frac{2x+1}{4}\cos(\pi x). \tag{5}$$

One should note that the set $[0,\infty)$ is invariant under *f*. Now recall that the Schwarzian derivative of a C^3 function is defined as

$$S(f)(x)=\frac{f'''(x)}{f'(x)}-\frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2. \tag{6}$$

It can be shown [2] that for *f* in Eq. 5, $S(f)(x)<0$ for every $x\in[0,\infty)$. This is very fortunate and reveals a lot of information about the basins of attraction of *f*. Specifically, it implies that the behavior of almost all (in the Lebesgue sense) of $[0,\infty)$ can be examined by looking at the ω-limit sets of its critical points [2].

Following the work of Chamberland, we now define the fixed points of *f* on $[0,\infty)$ as $0<\mu_1<\mu_2<\ldots$ and the critical points as $c_1<c_2<c_3<\ldots$. Then the following two relations hold: *i)* $n-1\le\mu_n\le n$ and *ii)* $\mu_n<c_n<\mu_{n+1}$. Chamberland also shows that $I_1=[0,\mu_1]$ and $I_2=[\mu_1,\mu_3]$ are invariant sets and that the only attracting orbits in I_2 are the cycles

$$A_1\equiv 1\to 2\to 1$$

$$A_2\equiv 1.192531907\ldots\to 2.138656335\ldots\to 1.192531907\ldots.$$

Since *f* has negative Schwarzian, this means that the interval I_2 can be partitioned as

$$I_2=B(A_1)\cup B(A_2)\cup N\,, \tag{7}$$

where $B(A_i)$ is the basin of attraction for A_i and *N* is a set of measure zero.

The interval $I_3=[\mu_3,\infty)$ is not an invariant set and therefore we can partition it as such [2]:

$$I_3=E_f\cup R_f\,, \tag{8}$$

where E_f is the set of points that leave I_3

$$E_f\equiv\{x\in I_3\,|\,\sup\omega(x)<\mu_3\}, \tag{9}$$

and R_f is the invariant subset of I_3

$$R_f\equiv\{x\in I_3\,|\,\inf\ \omega(x)\ge\mu_3\}. \tag{10}$$

Since the main conjecture claims that every integer $n\ge 3$ will eventually map to 1, this notation allows us to state the main conjecture as: $\mathbb{Z}^+\cap R_f=\varnothing$.

We can now further partition R_f to give

$$R_f=S_f\cup U_f\,, \tag{11}$$

where S_f is the open set consisting of all attracting periodic orbits of *f* in I_3 and their basins of attraction and U_f is the remaining closed set [2].

At this point we want to show that S_f contains all the nontrivial cycles of $f|_{\mathbb{Z}^+}$, but first we give a few lemmas.

Lemma 1: Suppose $p(x) = ax + b$ and $q(x) = cx + d$ are linear functions of x with $a,b,c,d \geq 0$ and $ad - cb \geq d - b$. If r is the composition of a permutation of n copies of p and m copies of q, then for every x,

$$q^m p^n(x) \leq r(x) \leq p^n q^m(x). \tag{12}$$

Proof: Suppose r is the composition of a permutation of n copies of p and m copies of q. Suppose $r \neq p^n q^m$, then there exists u and v that are compositions of permutations of p and q such that $r = uqpv$. Then from $ad - cb \geq d - b$, we get that for every x,

$$qp(x) = c(ax + b) + d \leq a(cx + d) + b = pq(x). \tag{13}$$

This in turn gives $qpv(x) \leq pqv(x)$. And since u is just a linear function with positive slope, we arrive at the conclusion that

$$r(x) = uqpv(x) \leq upqv(x) \leq \cdots \leq p^n q^m(x). \tag{14}$$

A similar argument provides the other inequality in Eq. 12.

Lemma 2: Suppose $p(x) = ax + b$, then

$$p^n(x) = a^n x + b\left(\frac{a^n - 1}{a - 1}\right). \tag{15}$$

Proof: Simple induction on n gives the desired result.

Theorem: Suppose $h\big|_{\mathbb{Z}^+} = T$ and its derivative is given by Eq. 3, then every periodic orbit of $h\big|_{\mathbb{Z}^+}$ is attracting.

Proof: Suppose $x_1 < x_2 < \cdots < x_n$ is a nonattracting periodic orbit on $\mathbb{Z}^+$, i.e. $h^n(x_1) = x_1$. As h essentially acts as either $p(x) = \tfrac{3}{2}x + \tfrac{1}{2}$ or $q(x) = \tfrac{1}{2}x$ on $\mathbb{Z}^+$, we can consider h^n as the composition of some permutation of l copies of p and k copies of q such that $l + k = n$. Notice that p and q satisfy the above requirements, so we can apply the lemmas and arrive at

$$q^k p^l(x_1) = \left(\tfrac{1}{2}\right)^k \left(\tfrac{3}{2}\right)^l x_1 + \left(\tfrac{1}{2}\right)^k \left(\left(\tfrac{3}{2}\right)^l - 1\right) \leq h^n(x_1) = x_1 \tag{16}$$

Now since our periodic orbit is nonattracting, we have that

$$\prod_{i=1}^{n} h'(x_i) = \left(\tfrac{1}{2}\right)^k \left(\tfrac{3}{2}\right)^l \geq 1. \tag{17}$$

Putting Eqs. 16 and 17 together along with the fact that $k, l \geq 1$, we arrive at the contradiction $h^n(x_1) > x_1$.

Corollary: For the *f* given in Eq. 5, any cycle of *f* on $\mathbf{Z}^+$ is an attracting periodic orbit and therefore S_f contains all the nontrivial cycles of $f|_{\mathbf{Z}^+}$.

Chamberland gives a shorter proof that specifically applies to the *f* defined in Eq. 5, however the preceding theorem is slightly more general in that it applies to all *f* that satisfy Eq. 3. A slightly stronger statement than *T* not having any nontrivial cycles on $\mathbf{Z}^+$ is then to conjecture that $S_f = \varnothing$. Recalling that *f* has negative Schwarzian, Chamberland extends this conjecture to say: Every critical point c_n has either $\omega(c_n) = A_1$ or $\omega(c_n) = A_2$. Numerical data supports this conjecture and shows that most critical points converge to A_1 while only a few actually converge to A_2.

At this point we can move on to study U_f. However, in this set-up, few results can actually be obtained. If we partition U_f into its respective bounded and unbounded parts

$$U_f = U_f^0 \cup U_f^\infty, \tag{18}$$

the only result worth mentioning here is that $\mathbf{Z}^+ \cap U_f^0 = \varnothing$ since every bounded orbit in $\mathbf{Z}^+$ is eventually periodic and therefore in S_f by our above theorem. This leaves us with the main conjecture as $S_f = \mathbf{Z}^+ \cap U_f^\infty = \varnothing$. Little has been done beyond this point for *f* satisfying Eq. 3.

§3 Functions With Critical Points at Each Integer

We now turn our attention towards *f* whose derivative vanishes on $\mathbf{Z}^+$. To do so, we extend Eq. 5 by adding two additional terms to give us

$$f(x) = x + \frac{1}{4} - \frac{2x+1}{4}\cos(\pi x) + \frac{1}{\pi}\left(\frac{1}{2} - \cos(\pi x)\right)\sin(\pi x) + h(x)\sin^2(\pi x). \tag{19}$$

The first extra term causes the derivative to vanish at any integer and the last term represents the freedom in our choice of *f*. Figure 1 shows the graphs of Eqs. 5 and 19 (for $h \equiv 0$). For the most part, they look identical. If we now consider *f* as a map on the complex plane, *f* is an entire holomorphic function if and only if the function *h* is entire and holomorphic. In fact, any entire holomorphic map that interpolates T in such a way that all integers are critical points is of the form of Eq. 19 [4]. So in general, we also have

$$f'(x)=\left[\tfrac{\pi}{2}\left(x+\tfrac{1}{2}\right)+2\sin(\pi x)+2\pi h(x)\cos(\pi x)+h'(x)\sin(\pi x)\right]\sin(\pi x)\,, \quad (20)$$

which not only shows that all integers are critical points, but also that there are other critical points determined by our choice of h.

Recall the definitions of the Fatou and Julia sets for a function $f:\mathbb{C}\to\mathbb{C}$. A point $z\in\mathbb{C}$ is in the Fatou set of f if there is a neighborhood U of z such that the sequence $\{f^n\}_{n=1}^{\infty}$ forms a normal family on U. The Julia set is then the complement of the Fatou set. Less formally, the Fatou set is the set of points that are "well-behaved" in the sense that nearby points will behave essentially the same way under repeated application of f, while the Julia set is far more chaotic. Typically, one also thinks of the Julia set as the boundary between the set of points that diverge nicely to infinity and those that converge to some bounded attracting orbit.

If we set $h\equiv 0$ in Eq. 19, we get that all the integers are in the Fatou set of f and that the connected components of the Fatou set can only be of four different types [4]:

i) (periodic) immediate basins of attracting periodic points;

ii) (periodic) domains at infinity in which the points diverge to ∞ locally uniformly;

iii) eventually periodic components, i.e. those which eventually map to one of the above types;

iv) wandering components, i.e. those whose forward orbits never repeat.

We also find that every Fatou component of f is simply connected [4]. Figure 2 shows a computer-generated image of the Fatou and Julia sets for $h\equiv 0$.

A consequence of the simple connectedness of every Fatou component is that no domain at infinity intersects the real line. To see why, suppose it did and there was a real x belonging to some unbounded domain U. Now all of x's orbit must lie in U, and since h preserves the reals, so does f. Therefore x must tend to ∞ along the real line. However, since all the numbers $\pm 2^n$ will obviously iterate to ± 1, they cannot be in U. Therefore U would have to be infinitely connected. This then contradicts the fact that every Fatou component is simply connected. Another argument shows that no domains at infinity can contain an integer, even for f and h that do not preserve the reals [4].

As a result, for the Fatou component of an integer, we can now rule out the second type in the possible Fatou component types. We would also like to rule out the fourth one, the wandering domains, as well. However, at this time only the following conjecture can be made: For some entire function h, the corresponding map f contains all the integers in its Fatou set and has no simply connected wandering domains intersecting the integers [4]. One should note that similar theorems exist for rational maps. However, proving such a conjecture would not show that all orbits on the integers eventually arrive at the trivial orbit, but it would show that all integer orbits are bounded.

Finally, since the critical points strongly influence the dynamics of our map, it would be beneficial to reduce the number of critical points. Ideally, one would like that the only critical points are the integers themselves. In light of Eq. 20, this involves solving the differential equation

$$2\sin(\pi x) + 2\pi h(x)\cos(\pi x) + h'(x)\sin(\pi x) = 0 . \tag{21}$$

Unfortunately, no such entire holomorphic solution h exists [4].

§4 Conclusion

The $3n+1$ problem has plagued mathematicians for over a half of a century. Although little progress has been made towards actually proving the conjecture, a lot of interesting math has been discovered along the way. Logicians have even tried to tackle this problem in the attempt to show that it is an unsolvable problem. In fact, the irony of the situation is that as evidence of its insolvability steadily grows, so do raw numerical calculations verifying its validity [1]. Currently, the conjecture has been checked for all integers up to 1.26×10^{16} and it has been shown that if a nontrivial cycle existed, it must contain more than 10^{8} numbers [4]. When once asked about the Collatz conjecture, the famous mathematician Paul Erdös gave an apparently very true response: "Mathematics is not yet ready for such problems."

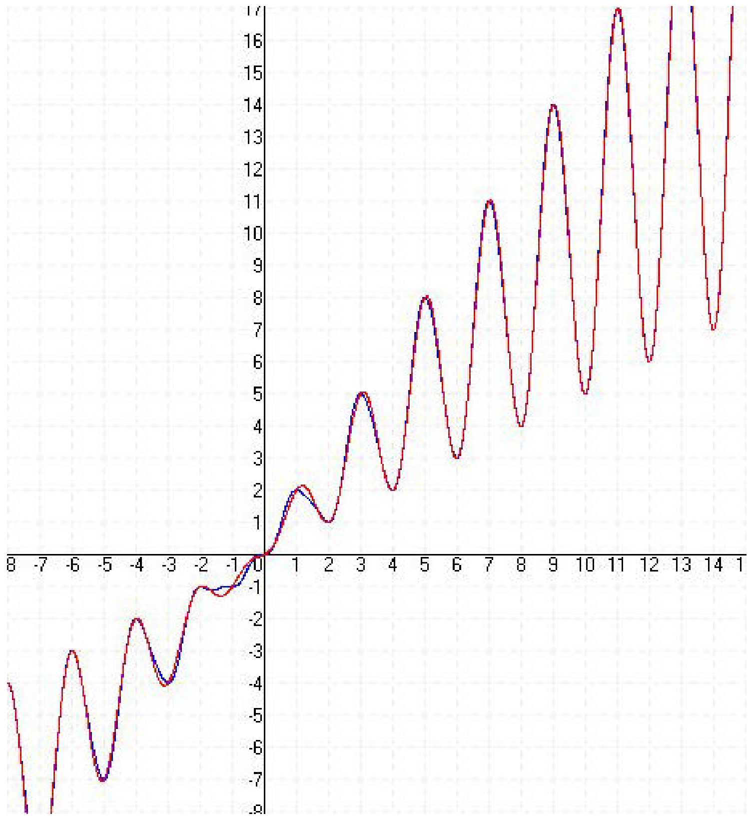

FIGURE 1: The red line shows the graph of Eq. 5, while the blue line shows Eq. 19 with $h \equiv 0$.

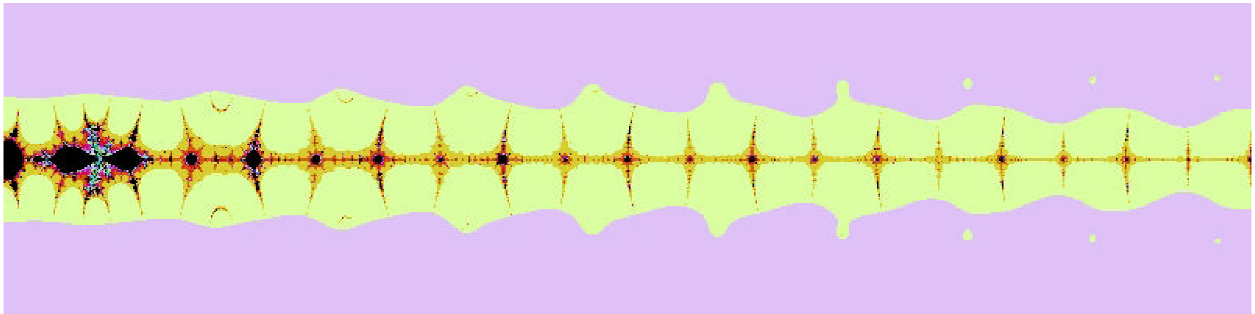

FIGURE 2: The Julia set for Eq. 19 with $h \equiv 0$. The horizontal axis represents the real part of z and runs from 0 to 20. Coloring is based on how long it takes for a point to iterate beyond a modulus of 2025. One can notice the large anomalies at each integer.

References

[1] E. Belaga, "Reflecting on the 3*x*+1 Mystery. Outline of a scenario - improbable or realistic?", 1998, preprint. Available at http://citeseer.nj.nec.com/belaga98reflecting.html

[2] M. Chamberland, "A continuous extension of the 3*x*+1 problems to the real line", *Dynam. Contin. Discrete Impuls. Systems* **2**:4 (1996), 495–509.

[3] J. C. Lagarias, "The 3*x*+1 problem and its generalizations", *Amer. Math. Monthly* **92**:1 (1985), 3–23.

[4] S. Letherman, D. Schleicher, and R. Wood, "The 3n+1 problem and holomorphic dynamics", *Experimental Mathematics* **8** (1999) 241–251.